Geometric Analysis with a view to data

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Geometric Analysis “Defined”

My (slightly) idiosyncratic view:

- Not simply analysis on manifolds, instead Geometric measure theory and variational analysis plus pieces of PDE, harmonic analysis, nonlinear functional analysis.
- My interests: geometry and analysis of sets, functions and measures in low and high dimensions. Includes things like the concentration of measure phenomenon.
- And: Do all this with a view to the illumination of very challenging data problems

In the rest of these slides I look at two examples of geometric analysis following form this viewpoint.

Exact Solutions of the L1TV functional

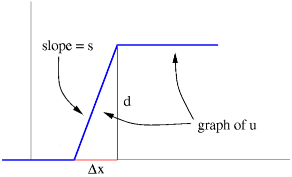
The L^1TV functional:

$$F(u) \equiv \int |\nabla u| dx + \lambda \int |u - d| dx$$

- **Not strictly convex:** $F(u)$ is not strictly convex $\Rightarrow$ we do not have uniqueness!
- **Homogeneity:** u is a minimizer for $d \longrightarrow Cu$ is a minimizer for Cd
- **Existence:** Since $TV(u)$ is lower semi-continuous in L^1, $F(u)$ is convex and coercive.

Intuition for the TV seminorm

Consider $F(u) \equiv \int |\nabla u|^p dx$



$$F(u) = s^p(\Delta x) = \frac{(s\Delta x)^p}{(\Delta x)^{p-1}} = d^p(\Delta x)^{1-p}$$

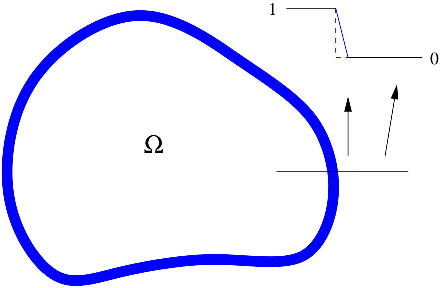
$(p > 1)$ $F(u) \underset{\Delta x \to 0}{\longrightarrow} \infty$ discontinuities are avoided: smooth u preferred,

$(p < 1)$ $F(u) \underset{\Delta x \to 0}{\longrightarrow} 0$ discontinuities cost nothing: step u preferred,

$(p = 1)$ $F(u) = d$ only jump magnitude “counts”, no bias towards smooth or step.

TV(χ_Ω)

Suppose u is a characteristic function of a set Ω? Can we see what TV(u) will be?

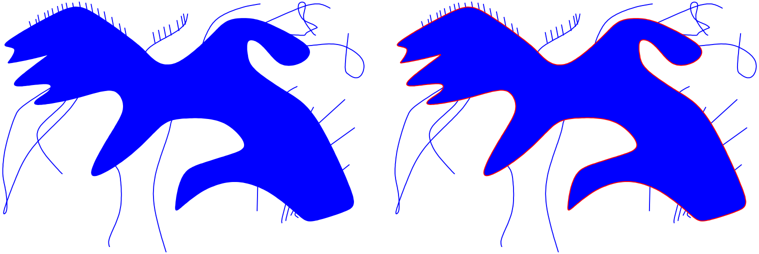


Using the figure of an approximate characteristic function, we can convince ourselves that TV(u) is simply the length of the boundary of the set Ω.

Wild Ω?

Suppose Ω is really wild? The "argument" above depended on Ω being nice. What can we conclude about $\mathrm{TV}(\chi_\Omega)$ in this case?

- There is a set $\partial^*\Omega$ called *reduced boundary* of Ω that coincides with the boundary that test functions can see
- $\mathrm{TV}(\chi_\Omega)$ picks up the boundary that integration against smooth test functions "sees".
- (There are details we are sweeping under the rug!)



Back to L^1TV: a useful property

L^1TV again:

$$F(u) \equiv \int |\nabla u| dx + \lambda \int |u - d| dx$$

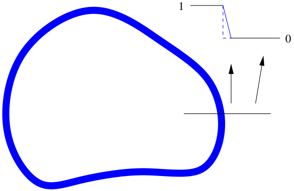
Chan and Esedoglu show that:

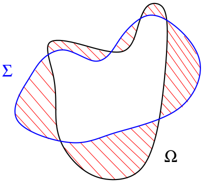
- $d = \chi_\Omega \Rightarrow$ for some Σ, $u = \chi_\Sigma$ is a minimizer.
- More Precisely: If u is any minimizer of $F(u)$ then for almost all $\mu \in [0, 1]$, $\chi_{\{x:u>\mu\}}$ is also a minimizer of $F(u)$, to get a minimizer that is a characteristic function.

What L^1TV reduces to for sets ...

$u = \chi_\Sigma$ **and** $d = \chi_\Omega \Rightarrow F(\Sigma) \equiv F(\chi_\Sigma) = \mathbf{Per}(\Sigma) + \lambda|\Sigma\Delta\Omega|$

- $u = \chi_\Sigma \longrightarrow$
  $\int |\nabla u| dx$ = perimeter of Σ
- $u = \chi_\Sigma, d = \chi_\Omega \longrightarrow$
  $\lambda \int |u - d| dx = \lambda \int |\chi_\Sigma - \chi_\Omega| dx = \lambda \text{Area}(\Sigma \triangle \Omega)$

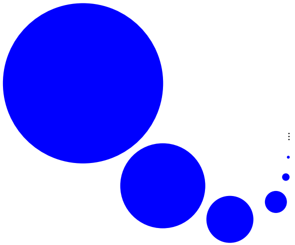




Example of Nonuniqueness

If Ω = $B_{\frac{2}{\lambda}}$ then $u = \alpha\chi_{B_{\frac{2}{\lambda}}}$ is a minimizer for any $\alpha \in [0, 1]$.

- We can concoct Ω's whose solutions $\Sigma(\lambda)$ have, as $\lambda \to \infty$, an infinite number of non-uniqueness points ...

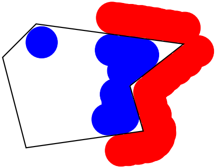


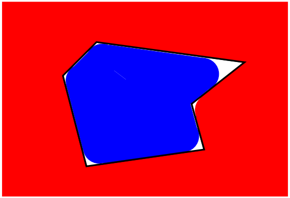
A Solution Envelope

Theorem

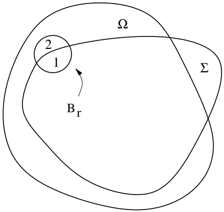
If $B_r \subset \Omega$ where $r \geq \frac{2}{\lambda}$, then $B_r \subset \Sigma$.

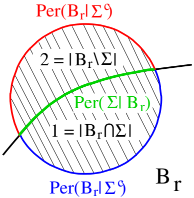
- edges are perfectly preserved if they can be touched by $\frac{2}{\lambda}$ balls in and out
- boundary of Σ is in the envelope between inside and outside $\frac{2}{\lambda}$ balls





Idea of Proof



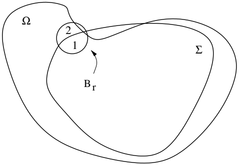


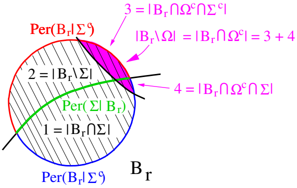
$$
\begin{aligned}
E(\Sigma \cup B_r) \quad - \quad & E(\Sigma) \\
= \quad & (\mathrm{Per}(B_r) - \lambda|B_r|) + (\lambda|B_r \cap \Sigma| - \mathrm{Per}(B_r \cap \Sigma)) \\
= \quad & \left(2\pi r - \frac{2}{R}\pi r^2\right) + \left(\frac{2}{R}\pi\rho^2 - 2\pi\rho^*\right) \\
= \quad & 2\pi r(1 - \frac{r}{R}) + 2\pi\rho(\frac{\rho}{R} - \frac{\rho^*}{\rho})
\end{aligned}
$$

Denoising shapes

Theorem

$$B_{\frac{2}{\lambda}} \underset{\sim}{\subset} \Omega \longrightarrow B_{\frac{2}{\lambda}-\epsilon} \subset \Sigma$$





$$E(\Sigma \cup B_r) - E(\Sigma) \leq 2\pi r(1 - \frac{r}{R}) + 2\pi\rho(\frac{\rho}{R} - \frac{\rho^*}{\rho}) + 2\lambda|B_r \setminus \Omega|$$

Discussion: $n > 2$ and Allard's Results

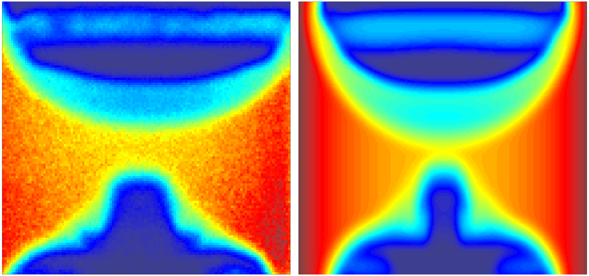
We use simpler means to arrive at some conclusions about minimizers of one functional in Allard's class:

- Results hold in $\mathbb{R}^n$: $\frac{2}{\lambda} \to \frac{n}{\lambda}$
- Allard gets a critical radius of $r = \frac{n-1}{\lambda}$, this is a local curvature. Our $r = \frac{n}{\lambda}$ is somehow global.
- We use the structure theorem for sets of finite perimeter, therefore n is unrestricted.
- Allard uses the more powerful regularity theory developed for minimal surface problems: his $n \leq 7$.

Symmetric Rearrangements

Idea: Use image geometry to generate robust measures.
Application: validation of simulation codes.

What is the distance between the following experiment and simulation?



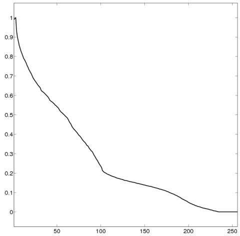
Invariant, Robust Measures

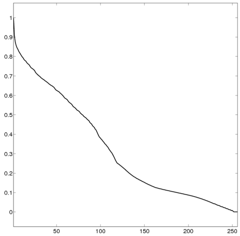
Idea: generate a rigid transformation invariant signature by looking at signatures derived from the *Steiner symmetrizations* or *symmetric decreasing rearangements*.

- **Symmetric decreasing rearrangements:** for f in $L^p(\mathbb{R}^n)$, the *symmetric decreasing rearrangement* f^* is the $L^p(\mathbb{R}^n)$ function such that $\{f^* \geq y\}$ is a disk centered at the origin in $\mathbb{R}^n$ such that $\mathcal{H}^n(\{f^* \geq y\}) = \mathcal{H}^n(\{f \geq y\})$. We will denote the mapping from f to f^* by $\mathcal{R}$.
- $||f||_p = ||f^*||_p$: Since L^p norms are integrals over areas of level sets.
- $||\nabla f||_p \geq ||\nabla f^*||_p$: used in applications to variational problems.
- $\mathcal{R}$ is *not* continuous in $W^{1,p}$ ...

Simplest Use: Areas

Easiest implementation: use the areas of the disks as a function of height of the disk (f^*) as a signature.





Now compare signatures between images.

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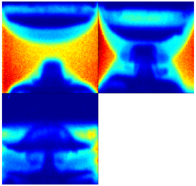
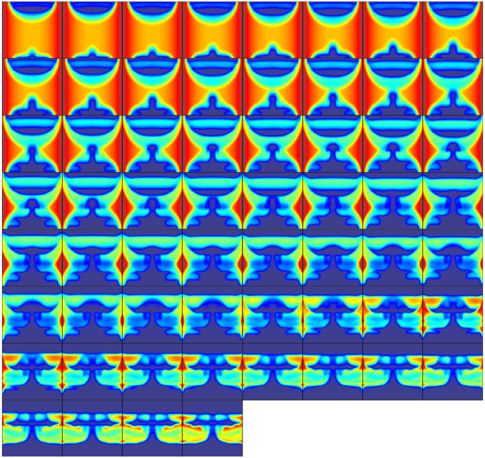
Computing and using these area signatures

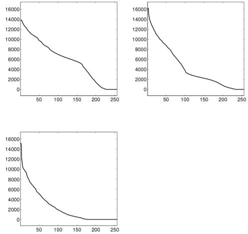
- **Regularize:** run *mean curvature flow* a bit on the simulated images and the experimental images.
- **Compute area signatures:** compute areas of the level sets.
- **1-D Warping:** use a simple warping method to compare the 1-D signatures.

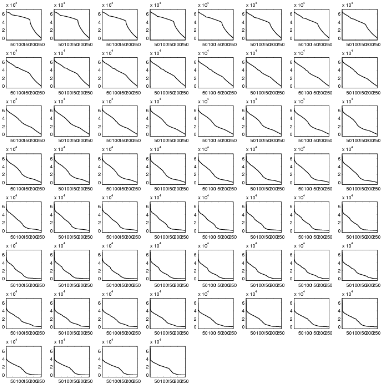
The experimental and simulated images





Area signatures





Results: distance curves

The registration problem is in fact a very good test of quality of the metric – validated by expert judgement